# Numberskull

Numberskull is a new puzzle, developed by David Wolstenholme of Topaccolades Limited, with characteristics that give it the potential to be a future classic

- Easy to understand
- Looks easy to solve, but its complexity and difficulty soon become apparent
- Its unique solution can always be reached by applying some simple techniques
- Enjoyable and challenging, requiring both arithmetic and logic
- Can be solved during a typical commuting journey
- Cross-cultural – no language barrier
- Compact - for publishing the puzzle and the solution

A Numberskull puzzle consists of a grid with 6 = signs and 6 whole number results already on it, together with 9 given whole numbers, normally placed in the bottom-right corner of the grid, all different. The puzzler simply has to fill in the grid by placing each of the nine numbers into the nine white squares, one in each, and placing any one of the four basic arithmetic operators (**+**, **-**, **x**, ÷) into the grey squares, so that 6 correct equations are formed (3 in left-to-right rows, 3 in top-to-bottom columns).

**Numberskull Example**

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | = | 52 |
| | | | | | | |
| | | | | | = | 40 |
| | | | | | | |
| | | | | | = | 2 |
| = | | = | | = | 2 3 6<br>7 11 31<br>40 44 49 | |
| 14 | | 48 | | 38 | | |

**Numberskull Solution**

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | ÷ | 2 | + | 49 | = | 52 |
| - | | x | | ÷ | | |
| 3 | + | 44 | - | 7 | = | 40 |
| + | | - | | + | | |
| 11 | - | 40 | + | 31 | = | 2 |
| = | | = | | = | | |
| 14 | | 48 | | 38 | | |

Note that the first operation is always evaluated first and its result is used in the second operation to produce the overall result. So, 3 + 4 x 2 means (3 + 4) x 2, which simplifies to 7 x 2 and equals 14. It does not mean 3 + (4 x 2) which simplifies to 3 + 8 and equals 11.

There is a unique solution. This can be found a step at a time, with no guessing required, by using arithmetic to determine some of the possible ways of achieving the results with the given numbers and, where necessary, by then applying logic to determine more about the positions of the numbers.

Topaccolades also produce Numberskull puzzles in which some of the given numbers and operators are already placed on the grid. This can make the puzzle easier to solve.

Although Numberskull is clearly not part of the Sudoku family of puzzles (none of the given numbers occurs more than once for one thing), because the logic required to solve it is not dissimilar, Numberskull can be seen as a combination of Countdown's Numbers Game and Sudoku/Kakuro.

**Details of Numberskull**

The idea seems very simple, and the unwary puzzler might think that it is easy to solve. However, it soon becomes apparent that many of the target numbers can be achieved in very many ways. For example, the target 52 in the example puzzle can be achieved in 19 ways with the given numbers:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 6 | + | 40 | = | 52 |
| 6 | x | 2 | + | 40 | = | 52 |
| 2 | + | 6 | + | 44 | = | 52 |
| 2 | + | 44 | + | 6 | = | 52 |
| 6 | + | 2 | + | 44 | = | 52 |
| 6 | + | 44 | + | 2 | = | 52 |
| 44 | + | 2 | + | 6 | = | 52 |
| 44 | + | 6 | + | 2 | = | 52 |
| 6 | ÷ | 2 | + | 49 | = | 52 |
| 6 | - | 3 | + | 49 | = | 52 |
| 6 | + | 49 | - | 3 | = | 52 |
| 49 | - | 3 | + | 6 | = | 52 |
| 49 | + | 6 | - | 3 | = | 52 |
| 3 | x | 7 | + | 31 | = | 52 |
| 7 | x | 3 | + | 31 | = | 52 |
| 11 | - | 3 | + | 44 | = | 52 |
| 11 | + | 44 | - | 3 | = | 52 |
| 44 | - | 3 | + | 11 | = | 52 |
| 44 | + | 11 | - | 3 | = | 52 |

The target 2 can be achieved in 55 ways, while the target 40 can be achieved in 49 ways. We find that, on average, each target number can be achieved in about 20 ways at the beginning of a puzzle in which no numbers or operators are already fixed on the grid. What might have seemed to be a simple puzzle now starts to look extremely difficult.

However, Numberskull can be turned into a challenging but quite solvable puzzle by good design such that it can be solved without the need for any guessing of number positions. Numberskull puzzles cannot be generated by arbitrarily placing the numbers on the empty grid, inserting arithmetic operators and determining the results to be used, as this would not guarantee a unique solution and would almost certainly require the puzzler to guess the positions of numbers and see whether this works out.

Instead, we create Numberskull puzzles to ensure that they have unique solutions and that they can be solved a step at a time through a partial analysis of the ways in which the results can be achieved with the available numbers. This gives the puzzler initial information about the potential numbers that might appear in each square. The puzzler can then use logic to help determine the positioning of the numbers.

Although the puzzler might use various techniques to solve them, Numberskull puzzles are designed to be solved using a set of eight techniques. At each step, the puzzler should try to apply these in order, and should find one that takes him or her a step further towards the unique solution. The first four techniques involve looking only at individual lines (equations); the remainder involve looking at number squares across the grid and include considering the implications of the intersection of row and column equations. For example, if a square can take 4, 7 or 36 according to its row equation, and 4, 36, 39 or 42 according to its column equation, the square can therefore take only those numbers that are common to both, namely 4 or 36.

The techniques are described in detail in Appendix 1, together with examples. Here, a simple summary is given.

| Objective and overview | Techniques involving only one line | Description | Technique involving two or more lines | Description |
|---|---|---|---|---|
| Complete a line or lines, including operators (and eliminate the numbers contained as potential numbers for all other squares) | Technique 1 | Find a line whose result can be achieved in only one way with the available numbers | Technique 5 | Find a number square that can take only one number according to the intersection of the potential numbers for its row and column, and whose row or column (or both) result can be achieved in only one way with the unique common number in this position |
| Fix the value of a single number square (and eliminate this as a potential number for all other squares) | Technique 2 | Find a number square that occurs in a line whose result can be achieved in more than one way, but all ways contain the same number in this square | Technique 6 | Find a number square that can take only one number according to the intersection of the potential numbers for its row and column (but does not yield a single solution for the row or column) |
| Find two number squares that can take only the same two numbers, and eliminate these as potential numbers for all other squares | Technique 3 | Find two squares in a line whose result can be achieved in more than one way, but all ways have either of the same two numbers in these squares | Technique 7 | Find two number squares from anywhere on the grid that can both take only the same two numbers, as determined either by their row or column or by the intersection of these |
| Find three number squares that can take only the same three numbers between them, and eliminate these as potential numbers for all other squares | Technique 4 | Find a line whose result can be achieved in more than one way, but all ways have one of the same three numbers in every number square | Technique 8 | Find three number squares from anywhere on the grid that can take, between them, only the same three numbers, as determined either by their row or column or by the intersection of these |

Technique 1 is by far the most important and common technique used in Numberskull puzzles. All Numberskull puzzles require the use of this technique, and the simplest puzzles can be solved using only this technique.

Other techniques could be used by the puzzler, e.g the puzzler might notice four squares that between them can take only four numbers and eliminate these as potential numbers from other squares, but the puzzles are not designed to require the use of any other techniques.

**Solving the example Numberskull puzzle**

Solving the example Numberskull puzzle using the approach and techniques outlined above results in the following steps:

**Step 1:**

Attempting Technique 1 for each row then each column, we find at least two ways of achieving each of the results 52, 40, 2, 14 and 48, so these are not unique. However, we can find only one way of achieving the result 38 with these numbers:

| | | | | | | |
|---|---|---|---|---|---|---|
| 49 | ÷ | 7 | + | 31 | = | 38 |

So, we fix the right column to these values.

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 49 | = | 52 |
| | | | | ÷ | | |
| | | | | 7 | = | 40 |
| | | | | + | | |
| | | | | 31 | = | 2 |
| = | | = | | = | | |
| 14 | | 48 | | 38 | | |

Right column fixed to:

| | | | | | | |
|---|---|---|---|---|---|---|
| 49 | ÷ | 7 | + | 31 | = | 38 |

**Step 2:**

Attempting Technique 1, we find no incomplete line with a unique solution, so try Technique 2. We find that the top row, result 52, can now be achieved in only 2 ways now that the 7, 31 and 49 have been fixed:

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | ÷ | 2 | + | 49 | = | 52 |
| 6 | - | 3 | + | 49 | = | 52 |

Both of these have 6 as the first number, so we can fix the top-left square to 6.

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | | | | 49 | = | 52 |
| | | | | ÷ | | |
| | | | | 7 | = | 40 |
| | | | | + | | |
| | | | | 31 | = | 2 |
| = | | = | | = | | |
| 14 | | 48 | | 38 | | |

Top-left number square fixed to 6

**Step 3:**

Attempting Techniques 1-4 on the individual lines, we find no lines that can be fixed, no squares that can be fixed, and no groups of 2 or 3 squares whose combined values can be defined and eliminated from other solutions.

We therefore attempt Technique 5, where we look at the intersection of rows and columns. We find that the top row has the two possible solutions identified in Step 2 above

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | ÷ | 2 | + | 49 | = | 52 |
| 6 | - | 3 | + | 49 | = | 52 |

while the middle column now has only six possible solutions (now that only 2,3,11, 40 and 44 are still available):

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 44 | - | 40 | = | 48 |
| 44 | x | 2 | - | 40 | = | 48 |
| 11 | - | 3 | + | 40 | = | 48 |
| 11 | + | 40 | - | 3 | = | 48 |
| 40 | - | 3 | + | 11 | = | 48 |
| 40 | - | 3 | + | 11 | = | 48 |

The first number position of the middle column can take 2, 11, 40 or 44. However, this is the same square as the middle position of the top row, which can take 2 or 3. This square must therefore take a 2 as this is the only number common to both.

Both the row and column have only one possible solution with a 2 in the relevant position, so we can fix both the row and column equations.

| | | | | | | |
|---|---|---|---|---|---|---|
| **6** | **÷** | **2** | **+** | **49** | **=** | **52** |
| | | **x** | | **÷** | | |
| | | **44** | | **7** | **=** | **40** |
| | | **-** | | **+** | | |
| | | **40** | | **31** | **=** | **2** |
| **=** | | **=** | | **=** | | |
| **14** | | **48** | | **38** | | |

Top row fixed to:

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | ÷ | 2 | + | 49 | = | 52 |

Middle column fixed to:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 44 | - | 40 | = | 48 |

With only two number squares not fixed and three lines incomplete, the solution becomes trivial and we can easily complete the puzzle using only Technique 1.

A fully-detailed description of the steps taken is given in Appendix 2. This highlights one of the key points to be aware of in relation to Numberskull, that only a partial analysis of the potential solutions is required. For example, in Appendix 2 we consider only 8 of the 19 possible solutions for the result 52, only 4 of the possible 55 ways for the result 2, and only 9 of the 49 possible ways for the result 40. At each step/technique, we carry out further analysis only if the technique looks as though it might yield a result, but as soon as we have found one or more additional solutions to prove that it will not, we move on. Of course, the puzzle simplifies considerably at each step. In the worked example, there are over 30 ways on average of solving each of the results in Step 1. This reduces to over 7 in Step 2, then 4 for Step 3. By the last couple of steps (not shown above), there is only 1 possible solution.

## House rules

The brief outline of Numberskull given at the beginning would cover puzzles with negative numbers, results of over 1000, and so on. We employ a number of ‘house rules’ that constrain the puzzles allowed in various ways, with the intention of making them more appealing and easier, including:

- The results should all be whole numbers in the range 1-100 (numbers people are generally most familiar with). Duplicates are permissible.
- The 9 given numbers, all different, should be whole numbers in the range 1-50.
- Up till, and including, the step in which the 6th number square is fixed, no alternatives should be found when employing the 8 techniques in order as described. So, for example, if Technique 1 is used and a line is found that has a unique solution so can be fixed, there should be no other line that can be fixed using Technique 1 at that step. Or, for example, if Technique 1 fails and Technique 2 is being attempted, only one number square should exist that can be fixed by considering only its row or column. Once 6 squares have been fixed, the solution generally becomes trivial and alternatives are allowed (and become inevitable).
- Any line that has more than 6 ways of achieving the result during a step should play no part in that step, either individually or to determine the potential values at an intersection of row and column. The squares within such a line may, of course, take part as a member of another line. This means that the puzzler does not need to carry out a full analysis for all lines in order to determine the intersection, if required, but can focus on lines with 6 or fewer potential solutions.

## Puzzle grading

A grading system is very useful for catering to puzzlers of different skills and experience. We have devised an internal grading system for Numberskull that includes two numbers, and takes the form T:A. T is a number from 1 to 8 and refers to the highest number technique required to solve the Numberskull puzzle using our recommended approach. A is a number (1-4) that refers to the complexity of the arithmetic used in the equations, as follows:

1. This is the default grading for an equation
2. This is a multiplication followed by a division where the multiplication has a result greater than 150, e.g. 18x28÷21=24
   We consider these to be a higher grade as most people are less familiar with handling such numbers
3. This is a division followed by a multiplication where the result of the division is not a whole number, e.g. 13÷3x6=26.
   When looking for solutions, these equations are far less obvious than others.
4. This is a division followed by a multiplication where the result of the division is not a whole number, and also the product of the first and last numbers is greater than 150, e.g. 18÷21x28=24
   This is essentially a combination of 2 and 3.

The A grade of the complete puzzle is that of the highest-graded equation.

For the public, we would probably simplify this grading system to 2 for T and 2 for A, or maybe just Easy, Moderate, Hard and Very Hard.

In addition, the more numbers and operators that are already fixed at the beginning, the simpler, in general, the puzzle.

**Solutions**

Solutions can be provided to puzzlers either with just the completed grid, as for the example, or together with the approach used to reach the solution. Since our Numberskull puzzles are designed to be solved in a certain way using the 8 techniques, attempted in order, and since we allow no alternatives until after the $6^{th}$ number square has been fixed, we can provide the puzzler with a unique solution code defining the techniques used and the numbers/lines fixed or restricted at each step up to that point.

So, for the example Numberskull, we can provide the following solution code defining the way to solve this:

C3(T1),6(T2:R1),R1+C2(T5)

This states that we first complete column 3 using technique T1, then fix the 6 using T2 operating on row 1, then complete row 1 and column 2 using T5.

The technique is always shown in brackets after the lines or numbers that were fixed or constrained (see next). In the case of T2, the single line involved is shown followed by a colon. For T5, if only one line is completed, the other line involved is also shown in this way.

If the solution code contains numbers between a pair of | symbols, this means that the squares containing these numbers were constrained to these numbers at that step and the numbers eliminated from other squares. For T3 and T4 the lines involved are always obvious, while for T7 and T8 they are not considered relevant.

So, in the solution code

|4,36,49|(T4),R1(T1),C2(T1),C1(T1)

the first step uses Technique 4 to constrain the squares containing the 4, 36 and 49.

**Software**

The generation of Numberskull puzzles with unique solutions is complex, particularly with the constraint of the house rule concerning no alternatives before the $6^{th}$ number is fixed. Topaccolades Ltd. have produced software that facilitates both the development and solving of puzzles and the production of puzzle pages.

The generation software allows us not only to produce puzzles from a given set of 9 numbers, but also to generate the set of 9 given numbers to be used in order to give us a particular opening Technique 1 equation.

**Intellectual Property**

Topaccolades has registered the name Numberskull as a trade mark in the UK.
Topaccolades Limited own the web domains: www.numberskull.com and www.numberskull.co.uk

**Appendix 1 - Techniques**

**Technique 1**:

Look for an incomplete equation for which there is only one way of achieving the result. The complete line can be fixed to the appropriate values.

Example: Given the nine numbers 2,3,6,7,11,31,40,44,49, suppose the result to be achieved is 38. This can be achieved in only one way:

| | | | | | | |
|---|---|---|---|---|---|---|
| 49 | ÷ | 7 | + | 31 | = | 38 |

**Technique 2:**

Look for an incomplete equation where there are two or more ways of achieving the result, but in all of these cases one square takes the same number. That square MUST therefore contain that number, so that number can be placed in that square on the grid.

Example: Given the nine numbers 1,5,6,7,15,17,29,40,45, suppose the result to be achieved is 96. This can be achieved in several ways:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | + | 15 | x | 6 | = | 96 |
| 15 | + | 1 | x | 6 | = | 96 |
| 17 | - | 1 | x | 6 | = | 96 |
| 45 | - | 29 | x | 6 | = | 96 |

However, the 3rd number is always a 6, so the square for the 3rd number must contain the 6.

**Technique 3:**

Look for an incomplete equation where there are two or more ways of achieving the result, but in all of these cases there are two squares that can each take only the same two numbers. Between them, these two squares MUST take these two numbers. Even though we cannot, at this stage, say which of these two eventually takes which number, we now know:

1) These two squares can take only these two numbers
2) All other squares cannot take these two numbers

This information can help us towards our overall solution.

Example: Given the nine numbers 1,2,3,4,5,6,8,13,33, suppose the result to be achieved is 85. This can be achieved in four ways only:

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | + | 13 | x | 5 | = | 85 |
| 13 | + | 4 | x | 5 | = | 85 |
| 4 | x | 13 | + | 33 | = | 85 |
| 13 | x | 4 | + | 33 | = | 85 |

However, the 1st and 2nd numbers are always either a 4 or 13, so between them they must contain the 4 and 13. 4 and 13 can now be eliminated as possible numbers for all other squares.

**Technique 4:**

Look for an incomplete equation where there are three or more ways of achieving the result, but the same three numbers are used in all the possible ways. Between them, these three squares MUST take these three numbers. Even though we cannot, at this stage, say which of these three eventually takes which number, we now know:

1) These three squares can take only these three numbers
2) All other squares cannot take these three numbers

This information can help us towards our overall solution.

Example: Given the nine numbers 1,4,9,10,15,27,31,39,45, suppose the result to be achieved is 83. This can be achieved in four ways only:

| | | | | | | |
|---|---|---|---|---|---|---|
| 39 | - | 1 | + | 45 | = | 83 |
| 39 | + | 45 | - | 1 | = | 83 |
| 45 | - | 1 | + | 39 | = | 83 |
| 45 | + | 39 | - | 1 | = | 83 |

Between them, the three number squares can take only the values 1, 39 and 45, so these numbers can be eliminated as potential values from all other squares. The potential values for the 1st number square of this line can be constrained to 39 or 45, while the 2nd and 3rd can both be constrained to 1, 39 or 45.

**Technique 5:**

Look for an empty number square where the possible values for its row equation have only one number in common with the possible values for its column equation and where the row or column (or maybe both) has only one way of achieving its result with this common value in this number square. The number square can be fixed to this common value, and the row or column with the resulting unique equation (or both) can be completed.

Example: Given the nine numbers 2,3,5,6,16,20,33,47,49, suppose the result to be achieved in the bottom row is 89 and the result for the middle column is 92. The 89 can be achieved in several ways:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 47 | - | 5 | = | 89 |
| 47 | x | 2 | - | 5 | = | 89 |
| 2 | x | 20 | + | 49 | = | 89 |
| 20 | x | 2 | + | 49 | = | 89 |

as can the 92:

| | | | | | | |
|---|---|---|---|---|---|---|
| 49 | - | 3 | x | 2 | = | 92 |
| 2 | x | 49 | - | 6 | = | 92 |
| 49 | x | 2 | - | 6 | = | 92 |
| 3 | x | 47 | - | 49 | = | 92 |
| 47 | x | 3 | - | 49 | = | 92 |

The potential values for the shared square are 2,20 or 47 for the row (its 2nd number) and 2,6 or 49 for the column (its 3rd number). As these have only one number in common, 2, this square must take 2. Only one of the equations for the column has a 2 in this position

| | | | | | | |
|---|---|---|---|---|---|---|
| 49 | - | 3 | x | 2 | = | 92 |

so the column can be fixed to these values. However, there are two equations with 2 in the 2nd place for the row, so this cannot be fixed.

**Technique 6:**

Look for an empty number square where the possible values for its row equation have only one number in common with the possible values for its column equation but where this common value does not yield a unique result in either direction. The number square can be fixed to this common value.

Example: Given the nine numbers 2,3,5,10,12,20,26,31,45, suppose the result to be achieved in the top row is 71 and the result for the right column is 82. The 71 can be achieved in several ways:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 20 | + | 31 | = | 71 |
| 20 | x | 2 | + | 31 | = | 71 |
| 31 | - | 5 | + | 45 | = | 71 |
| 31 | + | 45 | - | 5 | = | 71 |
| 45 | - | 5 | + | 31 | = | 71 |
| 45 | + | 31 | - | 5 | = | 71 |

as can the 82:

| | | | | | | |
|---|---|---|---|---|---|---|
| 10 | + | 31 | x | 2 | = | 82 |
| 31 | + | 10 | x | 2 | = | 82 |
| 2 | x | 31 | + | 20 | = | 82 |
| 31 | x | 2 | + | 20 | = | 82 |

The potential values for the shared square are 5,31 or 45 for the row (its 3$^{rd}$ number) and 2,10 or 31 for the column (its 1$^{st}$ number). As these have only one number in common, 31, this square must take 31. However, in neither the row nor the column is there a unique equation with 31 in the shared position, so no complete line can be fixed, but the square can be fixed to 31.

**Technique 7:**

Look for two empty number squares from anywhere on the grid that can each take only the same two numbers. Between them, these two squares MUST take these two numbers. Even though we cannot, at this stage, say which of these two eventually takes which number, we now know:

1) These two squares can take only these two numbers
2) All other squares cannot take these two numbers

When looking for squares that can take only two numbers, we can look either at the possible numbers for the square according to its row or column individually, or at the possible numbers that result from considering the intersection of the row and column. Overall, you should consider the possible numbers once intersections have been considered, but sometimes this is not feasible and you may have to consider just the possible values for its row or column.

Example: Given the nine numbers 2,4,6,9,15,18,22,32,44, suppose the result to be achieved in one line is 73 and the result in another line is 79. The 73 can be achieved in several ways:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 32 | + | 9 | = | 73 |
| 32 | x | 2 | + | 9 | = | 73 |
| 2 | x | 44 | - | 15 | = | 73 |
| 44 | x | 2 | - | 15 | = | 73 |
| 4 | x | 22 | - | 15 | = | 73 |
| 22 | x | 4 | - | 15 | = | 73 |

as can the 79:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 44 | - | 9 | = | 79 |
| 44 | x | 2 | - | 9 | = | 79 |
| 2 | x | 32 | + | 15 | = | 79 |
| 32 | x | 2 | + | 15 | = | 79 |
| 4 | x | 22 | - | 9 | = | 79 |
| 22 | x | 4 | - | 9 | = | 79 |

So, even without considering any interactions of rows and columns, we can see that the 3rd number in both lines can be only a 9 or 15. Therefore, we can eliminate 9 and 15 as potential numbers for all other squares.

Example: Given the nine numbers 3,6,9,10,12,14,21,32,35, suppose the result to be achieved in the top row is 80 and the result to be achieved in the left column is 91. The 80 can be achieved in several ways:

| | | | | | | |
|---|---|---|---|---|---|---|
| 14 | - | 6 | x | 10 | = | 80 |
| 32 | ÷ | 14 | x | 35 | = | 80 |
| 32 | x | 35 | ÷ | 14 | = | 80 |
| 35 | ÷ | 14 | x | 32 | = | 80 |
| 35 | x | 32 | ÷ | 14 | = | 80 |

as can the 91:

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | x | 35 | - | 14 | = | 91 |
| 35 | x | 3 | - | 14 | = | 91 |
| 6 | x | 21 | - | 35 | = | 91 |
| 21 | x | 6 | - | 35 | = | 91 |
| 9 | x | 14 | - | 35 | = | 91 |
| 14 | x | 9 | - | 35 | = | 91 |

Looking at the 3rd number in the column (91), we see that it can take only 14 or 35, but no other squares can take only these two numbers when the lines are considered individually. However, if we look at the intersection of this particular row and column (the first number in both cases), we see that the only numbers common to both are 14 and 35, so this square can also take only 14 or 35. We have therefore identified two squares that can take only 14 or 35, so can eliminate these two numbers as possible values from all other squares.

**Technique 8:**

Look for three empty number squares from anywhere on the grid that, between them, can take only the same three numbers. Between them, these three squares MUST take these three numbers. Even though we cannot, at this stage, say which of these three eventually takes which number, we now know:

1) These three squares can take only these three numbers
2) All other squares cannot take these three numbers

When looking for squares that can that form one of the set of three, we can look either at the possible numbers for the square according to its row or column individually, or at the possible numbers that result from considering the intersection of the row and column. Overall, you should consider the possible numbers once intersections have been considered, but sometimes this is not feasible and you may have to consider just the possible values for its row or column.

Example: Given the nine numbers 4,14,19,20,23,27,35,40,42, suppose the result to be achieved in the bottom row is 80 and the result to be achieved in the middle column is 99. The 80 can be achieved in several ways:

| | | | | | | |
|---|---|---|---|---|---|---|
| 40 | - | 20 | x | 4 | = | 80 |
| 23 | - | 19 | x | 20 | = | 80 |
| 27 | - | 23 | x | 20 | = | 80 |

as can the 99:

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | x | 20 | + | 19 | = | 99 |
| 20 | x | 4 | + | 19 | = | 99 |
| 4 | x | 19 | + | 23 | = | 99 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 19 | x | 4 | + | 23 | = | 99 |

The first two numbers for the column (99) can take only 4, 19 or 20, while the last number for the row can be only 4 or 20. So, between these three squares, they must take the 4, 19 and 20, so these numbers can be eliminated as possible numbers for all other squares.

The above example did not necessitate looking at the interaction of a row and column, only at the possible numbers for individual lines. However, it may be necessary to look at the interactions in some cases when using this technique, as in the following example.

Example: Given the nine numbers 1,4,5,9,13,25,29,32,50, suppose the result to be achieve in the top row is 71 and the result for the right column is 99 (so they intersect at the 3rd number of the row and the 1st number of the column. The 71 can be achieved in several ways:

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | x | 25 | - | 29 | = | 71 |
| 25 | x | 4 | - | 29 | = | 71 |
| 25 | - | 4 | + | 50 | = | 71 |
| 25 | + | 50 | - | 4 | = | 71 |
| 50 | - | 4 | + | 25 | = | 71 |
| 50 | + | 25 | - | 4 | = | 71 |

as can the 99:

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | x | 25 | - | 1 | = | 99 |
| 25 | x | 4 | - | 1 | = | 99 |
| 4 | x | 32 | - | 29 | = | 99 |
| 32 | x | 4 | - | 29 | = | 99 |

Looking at the row, we can see that the first two squares can each take only 4, 25 or 50, whereas the 3rd square can take 4, 25, 50 or 29.

Looking at the column, we see that the 1st square can take only 4, 25 or 32. This means that when we consider the interaction of the row and column, the shared square can take only 4 or 25.

Altogether, then, the three squares of the top row can, between them, take only 4, 25 or 50, so these numbers can then be eliminated as possible numbers for all other squares.

**Appendix 2 - Fully worked Numberskull example**

To demonstrate the use of these techniques, we shall work through solving the example Numberskull puzzle.

***Step 1***

Try Technique 1

Here we are looking to see whether there is a line whose result can be achieved in only one way. Start off by partially analysing each of the lines:

52
Three of the numbers (44, 6 and 2) can simply be added together to make 52. This alone provides six ways of reaching the result:

| | | | | | | |
|---|---|---|---|---|---|---|
| 44 | + | 6 | + | 2 | = | 52 |
| 44 | + | 2 | + | 6 | = | 52 |
| 6 | + | 44 | + | 2 | = | 52 |
| 6 | + | 2 | + | 44 | = | 52 |
| 2 | + | 6 | + | 44 | = | 52 |
| 2 | + | 44 | + | 6 | = | 52 |

so we can quickly see that this does not have a single solution.

40
Three of the numbers (31, 6 and 3) can be added together to make 40, so, as for 52, this alone gives 6 equations so there is not a unique solution.

2
We can quickly see two ways of making 2: 7-3-2 and 7-2-3, so this does not have a unique solution.

14
We can quickly see two ways of making 14: 6+11-3 and 6-3+11, so no unique solution exists.

48
40, 6 and 2 can be added together to make 48, so, as with 52, no unique result exists.

38
Finding a solution for 38 is not easy, but we can find 49÷7+31. We can't find another way, so this way of achieving 38 is unique, and we can fix the column to have these values.

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 49 | = | 52 |
| | | | | ÷ | | |
| | | | | 7 | = | 40 |
| | | | | + | | |
| | | | | 31 | = | 2 |
| = | | = | | = | | |
| 14 | | 48 | | 38 | | |

***Step 2***

*Technique 1*

Given this new grid, which has fixed some numbers (and also eliminated them as possibilities from other squares), we again look for unique solutions.

52
The options we found in Step 1 are no longer applicable, as we now know that the 3rd number is 49, so we must look for solutions with 49 in this position. We see that 49 from 52 leaves 3, and we can make 3 in two ways, 6÷2 and 6-3, so we have at least two ways of achieving 52.

40
We quickly see two ways of achieving 40 with 7 as the 3rd number: 44+3-7 and 3+44-7, so this is not unique.

2
We see two ways of achieving 2 with 31 as the 3rd number: 11-40+31 and 44-11-31, so this is not unique.

14
The two ways we found of achieving 14 in step 1 are still applicable, so no unique solution exists.

48
The six ways of achieving 48 in step 1 are still applicable, so this does not have a unique result.

We therefore have no line with a unique result, so need to try Technique 2.

*Technique 2*

Now we are looking for a square that can only take one value according to its row or column.

52
When trying Technique 1, we found two ways of achieving 52:

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | ÷ | 2 | + | 49 | = | 52 |
| 6 | - | 3 | + | 49 | = | 52 |

Further searching for other ways proves fruitless, so we have only these two ways. The first number is 6 in both cases, so this can be fixed to 6.

| | | | | | | |
|---|---|---|---|---|---|---|
| **6** | | | | **49** | **=** | **52** |
| | | | | **÷** | | |
| | | | | **7** | **=** | **40** |
| | | | | **+** | | |
| | | | | **31** | **=** | **2** |
| **=** | | **=** | | **=** | | |
| **14** | | **48** | | **38** | | |

***Step 3***

*Technique 1*

52, 40, 2 and 14
The solutions we identified previously are still applicable, so these are still not unique.

48
With the 6 now fixed, the solutions identified previously are no longer applicable. Looking for further solutions, we find 11+40-3, 11-3+40, 40+11-3 and 40-3+11, so, again there is no unique solution.

With no unique solutions for any remaining line, we then try Technique 2.

*Technique 2*

The solutions we have already identified do not have a unique number in any position that is not yet fixed, so we can move on to Technique 3.

*Technique 3*

This involves finding two unfixed number squares in any line that can take only the same two numbers. We can ignore the line with result 52 as this has only one unfixed square.

40
The two solutions identified in Step 2, 44+3-7 and 3+44-7, do have the same two numbers in the first two positions (3 and 44), so this looks a possibility. We must therefore look to see whether there are any other solutions. We find 44-11+7, so this means that we don't always have the same two numbers in the first two number positions.

2
The previously identified solutions are still valid and do not have the same two numbers in two unfixed positions.

14
The two solutions identified in Step 2, 6-3+11 and 6+11-3, do have the same two numbers in the last two positions (3 and 11), so this looks a possibility. However, we identify another possible solution: 6÷2+11, which means that this isn't the case.

48
The previously identified solutions are still valid and do not have the same two numbers in two unfixed positions.

So, with no success from Technique 3 we move onto Technique 4

*Technique 4*

This is concerned with finding a line with three unfixed numbers, where the only possible solutions all contain only the same three numbers. Only the line with result 48 has three unfixed numbers, so this is the only line we need to consider.

48
The four possible solutions we identified when considering Technique 1 for Step 3, 11+40-3, 11-3+40, 40+11-3 and 40-3+11, do all have the same three numbers, so this looks a possibility. However, if we analyse further, we also find two other possible solutions, 2x44-40 and 44x2-40, so again we have no success and move on to Technique 5.

*Technique 5*

Here we have to look at the intersection of a row and column at a square that is not fixed.

Intersection of top row (result 52) and middle column (result 48)

We find that the top row has the two possible solutions identified in Step 2 above

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | ÷ | 2 | + | 49 | = | 52 |
| 6 | - | 3 | + | 49 | = | 52 |

while the middle column has the six possible solutions we have identified in Step 3, and we can find no more:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 44 | - | 40 | = | 48 |
| 44 | x | 2 | - | 40 | = | 48 |
| 11 | - | 3 | + | 40 | = | 48 |
| 11 | + | 40 | - | 3 | = | 48 |
| 40 | - | 3 | + | 11 | = | 48 |
| 40 | - | 3 | + | 11 | = | 48 |

The first number position can take 2, 11, 40 or 44. However, this is the same square as the middle position of the top row, which can take 2 or 3. This square must therefore take a 2 as this is the only number common to both.

Both the row and column have only one solution with a 2 in the relevant position, so both the row and column equations can be fixed to these equations:

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | ÷ | 2 | + | 49 | = | 52 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | x | 44 | - | 40 | = | 48 |

| | | | | | | |
|---|---|---|---|---|---|---|
| **6** | **÷** | **2** | **+** | **49** | **=** | **52** |
| | | **x** | | **÷** | | |
| | | **44** | | **7** | **=** | **40** |
| | | **-** | | **+** | | |
| | | **40** | | **31** | **=** | **2** |
| **=** | | **=** | | **=** | | |
| **14** | | **48** | | **38** | | |

***Step 4***

*Technique 1*

40
This has a unique solution, 3+44-7, so this column can be fixed. This also fixes the last remaining number as it is the only one left.

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | ÷ | 2 | + | 49 | = | 52 |
| | | x | | ÷ | | |
| 3 | + | 44 | - | 7 | = | 40 |
| | | - | | + | | |
| 11 | | 40 | | 31 | = | 2 |
| = | | = | | = | | |
| 14 | | 48 | | 38 | | |

Completion of the last two lines is, of course, simple, as all the numbers are fixed.

**Appendix 3 - More examples and solutions (including a few where some of the numbers and/or operators are already fixed)**

**Puzzles to be solved**

## Numberskull 1

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | = | 25 |
| | | | | | | |
| | | | | | = | 42 |
| | | | | | | |
| | | | | | = | 5 |
| = | | = | | = | 2 7 8<br>12 13 28<br>32 33 44 | |
| 15 | | 93 | | 9 | | |

Grade: 1:1

## Numberskull 2

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | = | 20 |
| | | | | + | | |
| | | | | | = | 65 |
| - | | | | | | |
| | | 6 | | | = | 2 |
| = | | = | | = | 1 4<br>9 13 18<br>27 48 50 | |
| 63 | | 53 | | 36 | | |

Grade: 2:1

## Numberskull 3

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | = | 48 |
| | | | | | | |
| | | | | | = | 96 |
| | | | | | | |
| | | | | | = | 27 |
| = | | = | | = | 5 6 8<br>9 13 15<br>16 20 39 | |
| 100 | | 84 | | 42 | | |

Grade: 3:1

## Numberskull 4

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | = | 35 |
| | | | | | | |
| | | | | | = | 90 |
| | | | | | | |
| | | | | | = | 30 |
| = | | = | | = | 5 6 7<br>8 10 12<br>20 22 40 | |
| 31 | | 62 | | 74 | | |

Grade: 4:4

## Numberskull 5

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | = | 71 |
| | | | | | | |
| | | | | | = | 38 |
| | | | | | | |
| | | | | | = | 55 |
| = | | = | | = | | |
| 93 | | 7 | | 19 | | |

| | | |
|---|---|---|
| 1 | 4 | 5 |
| 7 | 10 | 13 |
| 22 | 41 | 50 |

Grade: 5:1

## Numberskull 6

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | = | 60 |
| | | | | | | |
| | | | | | = | 68 |
| | | | | | | |
| | | | | | = | 24 |
| = | | = | | = | | |
| 43 | | 4 | | 75 | | |

| | | |
|---|---|---|
| 1 | 3 | 4 |
| 7 | 8 | 19 |
| 22 | 33 | 40 |

Grade: 6:4

## Numberskull 7

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | = | 62 |
| | | | | | | |
| | | | | | = | 33 |
| | | | | | | |
| | | | | | = | 23 |
| = | | = | | = | | |
| 87 | | 47 | | 58 | | |

| | | |
|---|---|---|
| 2 | 3 | 7 |
| 10 | 20 | 23 |
| 37 | 49 | 50 |

Grade: 7:1

## Numberskull 8

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 3 | = | 11 |
| + | | | | | | |
| | | | | | = | 8 |
| | | | | | | |
| | | | | | = | 17 |
| = | | = | | = | | |
| 66 | | 3 | | 21 | | |

| | | |
|---|---|---|
| | 4 | 6 |
| 7 | 14 | 28 |
| 38 | 41 | 45 |

Grade: 8:3

**Solutions**

## Numberskull 1

| | | | | | | |
|---|---|---|---|---|---|---|
| 44 | - | 32 | + | 13 | = | 25 |
| ÷ | | + | | - | | |
| 2 | + | 28 | + | 12 | = | 42 |
| - | | + | | + | | |
| 7 | + | 33 | ÷ | 8 | = | 5 |
| = | | = | | = | | |
| 15 | | 93 | | 9 | | |

C1(T1),R3(T1),C3(T1) ...

## Numberskull 2

| | | | | | | |
|---|---|---|---|---|---|---|
| 18 | - | 48 | + | 50 | = | 20 |
| x | | - | | + | | |
| 4 | + | 1 | x | 13 | = | 65 |
| - | | + | | - | | |
| 9 | x | 6 | ÷ | 27 | = | 2 |
| = | | = | | = | | |
| 63 | | 53 | | 36 | | |

9(T2:C1),C2(T1),R2(T1) ...

## Numberskull 3

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | - | 5 | x | 16 | = | 48 |
| x | | + | | + | | |
| 15 | x | 9 | - | 39 | = | 96 |
| - | | x | | - | | |
| 20 | - | 6 | + | 13 | = | 27 |
| = | | = | | = | | |
| 100 | | 84 | | 42 | | |

20(T2:C1),|15,9|(T3),C1(T1),R1(T1),R2(T1) ...

## Numberskull 4

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | + | 7 | + | 22 | = | 35 |
| + | | x | | + | | |
| 5 | x | 10 | + | 40 | = | 90 |
| + | | - | | + | | |
| 20 | ÷ | 8 | x | 12 | = | 30 |
| = | | = | | = | | |
| 31 | | 62 | | 74 | | |

|5,6,20|(T4),R2(T1),C2(T1),R3(T1) ...

## Numberskull 5

| | | | | | | |
|---|---|---|---|---|---|---|
| 22 | - | 1 | + | 50 | = | 71 |
| x | | - | | - | | |
| 4 | - | 7 | + | 41 | = | 38 |
| + | | + | | + | | |
| 5 | x | 13 | - | 10 | = | 55 |
| = | | = | | = | | |
| 93 | | 7 | | 19 | | |

C1(T5:R1),|1,50|(T3),R3(T1),C3(T1) ...

## Numberskull 6

| | | | | | | |
|---|---|---|---|---|---|---|
| 40 | ÷ | 22 | x | 33 | = | 60 |
| - | | - | | - | | |
| 4 | x | 19 | - | 8 | = | 68 |
| + | | + | | x | | |
| 7 | + | 1 | x | 3 | = | 24 |
| = | | = | | = | | |
| 43 | | 4 | | 75 | | |

8(T6),|4,19|(T3),C3(T1),R1(T1),C2(T1) ...

## Numberskull 7

| | | | | | | |
|---|---|---|---|---|---|---|
| 49 | + | 50 | - | 37 | = | 62 |
| - | | - | | + | | |
| 20 | - | 10 | + | 23 | = | 33 |
| x | | + | | - | | |
| 3 | x | 7 | + | 2 | = | 23 |
| = | | = | | = | | |
| 87 | | 47 | | 58 | | |

C1(T1),|10,50|(T7),C2(T1) ...

## Numberskull 8

| | | | | | | |
|---|---|---|---|---|---|---|
| 14 | - | 6 | + | 3 | = | 11 |
| + | | - | | ÷ | | |
| 45 | - | 41 | + | 4 | = | 8 |
| + | | + | | x | | |
| 7 | + | 38 | - | 28 | = | 17 |
| = | | = | | = | | |
| 66 | | 3 | | 21 | | |

|4,14,28|(T8),R1(T1),R2(T1) ...